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Hereditarily paracompact and compact monotonically normal spaces

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Abstract

We give a proof that every compact, hereditarily paracompact, monotonically normal space is the continuous image of a compact linearly ordered space. © 2001 Elsevier Science B.V. All rights reserved.

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Nikiel [4] has conjectured that every compact, monotonically normal space is the continuous image of a compact linearly ordered space. In [5] and [6] I proved:

Theorem 1. *Every separable compact monotonically normal space is the continuous image of a compact linearly ordered space.*

In compact monotonically normal spaces, separability implies first countability which implies countable tightness which is equivalent to hereditary paracompactness and also to having all closed sets be G_δ sets [1,2]. Knight announced that the *separability* in Theorem 1 could be replaced by *first countability*, a claim I have also frequently made, but neither of us has published a proof. Here I prove the strictly stronger:

Theorem 2. *Every hereditarily paracompact, compact, monotonically normal space is the continuous image of a compact linearly ordered space.*

Proof. Suppose X is countably tight, compact, and monotonically normal, and all closed sets are G_δ.

Since X is *monotonically normal* points are closed and for each $x \in X$ and open U with $x \in U$, there is an open $H(x, U)$ with $x \in H(x, U) \subset U$ such that

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(1) (*normality*) $H(x,U) \cap H(y,V) \neq \emptyset$ implies either $x \in V$ or $y \in U$.
(2) (*monotonicity*) If $x \in U \subset V$, $H(x,U) \subset H(x,V)$.
We use $H^2(x,U)$ for $H(x,H(x,U))$ and $H^3(x,U)$ for $H(x,H^2(x,U))$.
Since X is *countably tight*, if $A \subset X$, every limit point of A is a limit point of a countable subset of A.
If $C \subset X$ we use $\partial C = C \cap (\overline{X - C})$ for the *boundary* of C and C^0 for the *interior* of C. □

1. Breakdowns

(See [4,5] for similar concepts and lemmas.)
Let $\mathcal{C} = \{C \subset X \mid C$ is compact and $(*)$ and $(**)$ (below) hold$\}$.

$(*)$ *If $\{x_i \mid i < 3\} \subset C$ and $\{W_i \mid i < 3\}$ are open sets in X with disjoint closures and $x_i \in W_i$ for all $i < 3$, then $H^3(x_i, W_i) \subset C$ for some i.*

(In particular $|\partial C| \leqslant 2$.)

$(**)$ *If $\{x_0, x_1, y_0, y_1\} \subset C$ and $\{W_i \mid i < 2\}$ and $\{U_i \mid i < 2\}$ are each pairs of open sets with disjoint closures and $x_i \in W_i$ and $y_i \in U_i$ for each $i < 2$, then two of $H^3(x_0,W_0) \cap H^3(y_0,U_0)$, $H^3(x_0,W_0) \cap H^3(y_1,U_1)$, $H^3(x_1,W_1) \cap H^3(y_0,U_0)$, and $H^3(x_1,W_1) \cap H^3(y_1,U_1)$ are contained in C (and thus in C^0).*

Suppose $C \in \mathcal{C}$.
We define a *breakdown* of C to be a set

$$\mathcal{F} = \{\mathcal{F}_n \mid n \in \omega\}$$

where:
(0) For some n, each term of $\mathcal{F}_n$ contains at most one point of ∂C in its closure.
(1) Each $\mathcal{F}_n$ is a finite cover of C by open sets each of which meets C.
(2) If $F \in \bigcup \mathcal{F}$ and $|F \cap C| > 1$, there are $n \in \omega$ and $x \neq y$ in F such that $F' \in \mathcal{F}_n$ implies either $x \notin \overline{F}'$ or $y \notin \overline{F}'$.
(3) If $M \subset \omega$ is finite, there is an $n \in \omega$ such that $\mathcal{F}_n$ refines $\{H(x, G_M(x)) \mid x \in C\}$ where, for $x \in C$, $G_M(x) = \bigcap\{G \mid x \in G$ and, for some $m \in M$, either $G \in \mathcal{F}_m$ or $G = X - \overline{F}$ for some $F \in \mathcal{F}_m$ or $G = C^0\}$.
Suppose $\mathcal{F}$ is a breakdown of C. Define

$$\mathcal{K}(\mathcal{F}) = \left\{\bigcap_{n\in\omega} F_n \cap C \mid \forall n \in \omega,\ F_n \in \mathcal{F}_n \text{ and } \exists m \in \omega \text{ with } \overline{F}_m \subset F_n\right\}.$$

Then $\mathcal{K}(\mathcal{F})$ is a disjoint, compact cover of C.

If $K \in \mathcal{K}(\mathcal{F})$, (2) ensures $\partial K \neq \emptyset$ unless $|K| = 1$ and $K \in \bigcup \mathcal{F}$. For each pair $\langle F, G\rangle$ with $\{F, G\} \subset \bigcup \mathcal{F}$, if possible choose x_{FG} with $F \subset H(x_{FG}, G)$. Then the set of all x_{FG}'s is countable and, by (3), dense in $C - \bigcup\{K^0 \mid K \in \mathcal{K}$ and $K \neq K^0\}$. This set is thus separable since separability is hereditary in monotonically normal spaces [2].

Lemma 0. *If $K \in \mathcal{K}(\mathcal{F})$ and $P \subset C - K$ is finite, there are an open G in X and $x \in G$ with $P \cap G = \emptyset$ such that $K \subset H(x, G)$. So*:

Lemma 1. *If $K \neq K'$ in $\mathcal{K}(\mathcal{F})$, $\{x_0, x_1\} \subset K'$, and W_0 and W_1 are disjoint open sets with $x_0 \in W_0$ and $x_1 \in W_1$, then there is at most one $i < 2$ with $K \cap H(x_i, W_i) \neq \emptyset$.*

For $\langle F, F'\rangle$ from $\bigcup \mathcal{F}$, let $\mathcal{K}(F, F')$ be the set of all $K \in \mathcal{K}(\mathcal{F})$ such that $K \subset F \subset \overline{F} \subset F'$ and either

(1) there are $\{x_i \mid i < 3\} \subset K$ and open $\{W_i \mid i < 3\}$ in X with disjoint closures such that, for all $i < 3$, $x_i \in W_i$ and $H^3(x_i, W_i) \not\subset F'$, or

(2) there are $\{x_0, x_1, y_0, y_1\} \subset K$, open sets $\{W_i \mid i < 2\}$, with disjoint closures and open sets $\{U_i \mid i < 2\}$ with disjoint closures such that, for all $i < 2$, $x_i \in W_i$ and $y_i \in U_i$, and at least three of $H^3(x_0, W_0) \cap H^3(y_0, U_0)$, $H^3(x_0, W_0) \cap H^3(y_1, U_1)$, $H^3(x_1, W_1) \cap H^3(y_0, U_0)$, and $H^3(x_1, W_1) \cap H^3(y_1, U_1)$ are *not* contained in F'.

If $\mathcal{K}(F, F')$ is infinite there is an infinite $\mathcal{A} \subset \mathcal{K}(F, F')$ where one of (1) and (2) holds. Suppose (1) holds for all $K \in \mathcal{A}$ and for each $i < 3$ let x_{iK} and W_{iK} be the x_i and W_i for K in (1). By Ramsey's $\omega \to \omega_r^2$ [3] and Lemma 1, there are an $i < 3$ and infinite $\mathcal{B} \subset \mathcal{A}$ with $\{H^2(x_{iK}, W_{iK}) \mid K \in \mathcal{B}\}$ disjoint. Thus there is a limit point p of $\bigcup\{H^3(x_{iK}, W_{iK}) \mid K \in \mathcal{B}\}$ not in F' and in at most one $H^2(x_{iK}, W_{iK})$. Since all $x_i \in F \subset \overline{F} \subset F'$ we have a contradiction of H. Similarly, if (2) holds for some $K \in \mathcal{K}(F, F')$, and $K' \neq K$ in $\mathcal{K}(\mathcal{F})$, by Lemma 1 there can be at most one of the $H(x_i, W_i) \cap H(y_j, U_j)$ for K which intersects K'. So if there is an infinite $\mathcal{A} \subset \mathcal{K}(F, F')$ with (2) holding for all $K \in \mathcal{A}$, an exactly similar argument leads to a contradiction. Hence:

Lemma 2. *$\mathcal{K}(F, F')$ is finite for all $\langle F, F'\rangle$ from $\bigcup \mathcal{F}$. (Thus $\bigcup\{\mathcal{K}(F, F') | F$ and F' are in $\bigcup \mathcal{F}\}$ is countable.)*

We want to prove:

Lemma 3. *There is a breakdown $\mathcal{F}$ of C such that $\mathcal{K}(\mathcal{F}) \subset \mathcal{C}$.*

Proof. To this end, by induction on β, for each $\beta < \omega_1$ we choose a breakdown

$$\mathcal{F}(\beta) = \left\{\mathcal{F}_n(\beta) \mid n \in \omega\right\}$$

of C and let $\mathcal{K}(\beta)$ be the $\mathcal{K}$ associated with $\mathcal{F}(\beta)$.

Choose $\mathcal{F}(0)$ arbitrarily. For $\beta > 0$ the $\mathcal{F}(\beta)$ is chosen in such a way that $\alpha < \beta < \omega_1$ implies $\{\mathcal{F}_m(\alpha) \mid m \in \omega\} \subset \{\mathcal{F}_n(\beta) \mid n \in \omega\}$ and, for limit β, $\{\mathcal{F}_n(\beta) \mid n \in \omega\} = \{\mathcal{F}_m(\alpha) \mid m \in \omega,\ \alpha < \beta\}$ which is indeed a breakdown of C since each $\mathcal{F}(\alpha)$ for $\alpha < \beta$ is one.

So suppose $\beta = \alpha + 1$ and $\mathcal{F}(\alpha)$ has been chosen.

Suppose $K \in \mathcal{K}(\alpha) - \mathcal{C}$.

If $K \subset C^0$ there are associated F and F' in $\bigcup \mathcal{F}(\alpha)$ with $F' \subset C^0$ such that $K \in \mathcal{K}(F, F')$. Choose $\{x_i \mid i < 2\} \subset K$ and open $\{W_i \mid i < 2\}$ with $\overline{W}_0 \cap \overline{W}_1 = \emptyset$ such that, for all $i < 2$, $x_i \in W_i$ and $H^3(x_i, W_i) \not\subset F'$. Let $\mathcal{W}_K = \{X - \overline{W}_i \mid i < 2\}$.

There is at most one term of $\mathcal{K}(\alpha)$ containing a particular point of ∂C, and $|\partial C| \leqslant 2$. Since $C \in \mathcal{C}$ but $K \notin \mathcal{C}$ if $K \not\subset C^0$, there must be some F and F' in $\bigcup \mathcal{F}(\alpha)$ with

$K \subset F \subset \overline{F} \subset F'$, $\{x_i \mid i < 2\} \subset K$, and open $\{W_i \mid i < 2\}$ with $x_i \in W_i$ and $\overline{W}_0 \cap \overline{W}_1 \neq \emptyset$, such that $x_0 \in \partial C$ and $H^3(x_1, W_1) \not\subset F'$. Let

$$\mathcal{W}_K = \{W_0, X - \overline{H(x_0, W_0)}\}.$$

Choose $\mathcal{F}(\beta)$ to be a breakdown of C containing $\mathcal{F}(\alpha)$ and $\{\mathcal{W}_K \mid K \in (\mathcal{K}(\alpha) - \mathcal{C})\}$. Since $\mathcal{F}(\alpha)$ is a breakdown of C and $\mathcal{K}(\alpha) - \mathcal{C}$ is countable, there is such a breakdown.

If there is some $\beta < \omega_1$ with $\mathcal{K}(\beta) - \mathcal{C} = \emptyset$, Lemma 3 holds. Otherwise, for all β we can choose $K(\beta) \in \mathcal{K}(\beta) - \mathcal{C}$. For each β there are $F(\beta)$ and $F'(\beta) \in \bigcup \mathcal{F}(\beta)$ associated with $K(\beta)$ and, by the pressing down lemma, we can choose an uncountable T from the limit in ω_1 and an F and F' such that $F = F(\beta)$ and $F' = F'(\beta)$ for all $\beta \in T$.

If $F' \not\subset C^0$, $K(\beta) \cap \partial C \neq \emptyset$ for any $\beta \in T$. Since $|\partial C| \leqslant 2$ we can assume that the $x_{0\beta}$'s (for $\{x_{0\beta}\} = K(\beta) \cap \partial C$)) are the same for all $\beta \in T$. Since $\mathcal{W}_{K(\alpha)} = \{W_{0\alpha}, X - \overline{H(x_{0\alpha}, W_{0\alpha})}\}$, for $\alpha < \beta$ in T, $x_{\beta 0} = x_{\alpha 0}$ and $K_\beta \subset K_\alpha$ so $K_\beta \subset W_{0\alpha}$. Thus $K_\beta \cap W_{1\alpha} = \emptyset$.

If instead $F' \subset C^0$, then $K(\beta) \in \mathcal{K}(F, F')$ for all $\beta \in T$, and we can use exactly the same proofs used for Lemmas 1 and 2 to show that any set of *disjoint* $K(\beta)$'s for $\beta \in T$ is finite. If $\alpha < \beta$ in T put $\{\alpha, \beta\}$ into pot I if $K(\beta) \subset K(\alpha)$ (which happens if $K(\alpha) \cap K(\beta) \neq \emptyset$) and put $\{\alpha, \beta\}$ into pot II if $K(\alpha) \cap K(\beta) = \emptyset$. Then the partition theorem of Erdös [3] that $\omega_1 \to (\omega_1, \omega)^2$ assures us that, since there is no infinite set of disjoint $K(\beta)$'s with $\beta \in T$, there must be an uncountable $S \subset T$ such that for all $\alpha < \beta$ in S, $K(\beta) \subset K(\alpha)$. By our choice of $x_{0\alpha}$ and $x_{1\alpha}$ for $\alpha \in S$ and the fact that $\mathcal{W}_{K(\alpha)} = \{X - \overline{W}_{i\alpha} \mid i < 2\}$ for every $\alpha \in S$ there is an $i_\alpha \in 2$ such that, for all $\beta > \alpha$ in S, $K(\beta) \cap W_{i\alpha} = \emptyset$. There is some $i \in 2$ such that $\{\alpha \in S \mid i_\alpha = i\}$ is uncountable.

Hence in all cases there are an uncountable $R \subset T$ and an $i \in 2$ such that, for all $\alpha < \beta$ in R, $K(\beta) \subset K(\alpha)$ and $K(\beta) \cap W_{i\alpha} = \emptyset$. We again apply $\omega_1 \to (\omega_1, \omega)^2$ and, if $\alpha < \beta$ in R, put $\{\alpha, \beta\}$ in pot I if $x_{i\alpha} \in H(x_{i\beta}, W_{i\beta})$ and put $\{\alpha, \beta\}$ into pot II otherwise. Recall that for all $\beta \in R$, $(H^3(x_{i\beta}, W_{i\beta}) - F') \neq \emptyset$. If there is an infinite set $Q \subset R$ all of whose pairs are in pot II, then $\{H^2(x_{i\beta}, W_{i\beta}) \mid \beta \in Q\}$ are disjoint and, by the proof of Lemma 2, this is impossible. Thus there is an uncountable $Q \subset R$ such that for all $\alpha < \beta$ in Q, $x_{i\alpha} \in H(x_{i\beta}, W_{i\beta})$.

Choose $x \in (\bigcap_{\beta \in Q} K(\beta)) \cap \overline{\{x_{i\alpha} \mid \alpha \in Q\}}$. Since X has countable tightness there is $\beta \in Q$ such that $x \in \overline{\{x_{i\alpha} \mid \alpha < \beta \text{ in } Q\}}$. Since $\{x_{i\alpha} \mid \alpha < \beta \text{ in } Q\} \subset H(x_{i\beta}, W_{i\beta})$, $x \in \overline{H(x_{i\beta}, W_{i\beta})} \subset W_{i\beta}$. If $\gamma > \beta$ in Q, $K(\gamma) \cap W_{i\beta} = \emptyset$. But this contradicts $x \in K(\gamma)$ for all $\gamma \in Q$.

This proves Lemma 3 and we can choose a particular breakdown $\mathcal{F}(C)$ for which $\mathcal{K}(C) = \mathcal{K}(\mathcal{F}(C)) \subset \mathcal{C}$. □

2. The tree T

By induction on ω_1 we build a tree $T = \bigcup_{\beta \leqslant \omega_1} T_\beta$ as well as T^*_β for each $\beta < \omega_1$, as follows.

(1) $T_0 = \{X\} = T^*_0$.

(2) For all $0 < \beta < \omega_1$, T_β is a set of disjoint members of $\mathcal{C}$ and $T_\beta^* = \{K \in T_\beta \mid K^0 \neq \emptyset$ and $\partial K \neq \emptyset\}$.

(3) If $\beta = \alpha + 1$, let $T_\beta = \bigcup\{\mathcal{K}(C) \mid C \in T_\alpha^*\}$.

(5) If β is a limit $\leqslant \omega_1$, define $T_\beta = \{\bigcap_{\alpha<\beta} C_\alpha \mid C_\alpha \in T_\alpha$ and $(\bigcap_{\alpha<\beta} C_\alpha) \neq \emptyset\}$.

Note that if $\bigcup_{\alpha<\beta} T_\alpha \subset \mathcal{C}$, each term of T_β, as defined, is automatically in $\mathcal{C}$. Thus T automatically satisfies (2) and is well defined. Also observe that if β is a limit ordinal and $K \in T_\beta$, then $\partial K \neq \emptyset$. If $C \in \mathcal{C}$ and $K \in \mathcal{K}(C)$, then by condition (2) of the definition of a breakdown, $\partial K - \partial C \neq \emptyset$ unless $|\partial K| = 1$.

Saturation Lemma. *If $p \in V$ which is open in X and $S = \{K \in T \mid p \notin K$ and $K - V \neq \emptyset\}$, there is an open Z with $p \in Z$ and $Z \cap (\bigcup S) = \emptyset$.*

Proof. We claim that for all $q \in X - V$ there are disjoint open neighborhoods Z_q of p and U_q of q such that $q \notin K \in S$ implies either $K \cap Z_q = \emptyset$ or $K \cap U_q = \emptyset$. This claim would prove the lemma for there would be a finite $Q \subset (X - V)$ such that $\{U_q \mid q \in Q\}$ covers $X - V$. Then $Z = \bigcap\{Z_q \mid q \in Q\} - \bigcup\{K \in S \mid K \cap Q \neq \emptyset\}$ would satisfy the lemmas requirements.

So suppose $q \in X - V$ and let $S' = \{K \in S \mid q \notin K\}$. Since all closed sets are G_δ, there is a maximal $\alpha < \omega_1$ with p and q in the same term C of T_α. Define $Z_0 = H^2(p, V)$ and $U_0 = H(q, X - \overline{H(p, V)})$. Take $Z_q = Z_0$ and $U_q = U_0$ unless $C \in T_\alpha^*$ in which case $p \in A \in \mathcal{K}(C)$ and $q \in B \in \mathcal{K}(C)$ and there are disjoint open $Z_1 \supset A$ and $U_1 \supset B$ and we let $U_q = U_0 \cap H(q, U_1)$ and $Z_q = Z_0 \cap H(p, Z_1)$.

Suppose $K \in S'$. There is a maximal $\beta \leqslant \alpha$ such that p, q, and K are all in the same term C' of T_β; let K' be the term of $\mathcal{K}(C')$ containing K. If $\beta < \alpha$, which is the case if $C \notin T_\alpha^*$, the term K^* of T_β containing both p and q is in $\mathcal{K}(C')$. By Lemma 1 there is an open G and $x \in G$ with $K' \subset H(x, G)$ and $G \cap K^* = \emptyset$. So at most one of U_0 and Z_0 intersects K' (or K). If $\beta = \alpha$, then $C = C'$, $C \in T_\alpha^*$, and K' cannot be both A and B; say $K' \neq A$. If $K' = B$, then $K' \cap Z_q = \emptyset$. If $K' \neq B$, then by Lemma 0 there is an open G and $x \in G$ with $K' \subset H(x, G)$ so that neither p nor q is in G. Thus, at most of $H(q, U_1)$ and $H(p, Z_1)$ can intersect $K' \supset K$. $\square$

3. T^2 and C^2

Define

$$T^2 = \{K \in T \mid |\partial K| = 2\}.$$

If $K \in T^2$ and $\partial K = \{x_0, x_1\}$, choose open W_0 and W_1 with disjoint closures so $x_0 \in W_0$ and $x_1 \in W_1$, and define $V_i(K) = H^5(x_i, W_i)$ for each $i < 2$. If $C \in T$, define

$$C^2 = \{K \in T^2 \mid K \subset C \subset (K \cup V_0(K) \cup V_1(K)) \text{ and } V_i(K) \not\subset C^0 \text{ for any } i < 2\}.$$

Observe that $X^2 = \emptyset$.

Lemma 4. *Suppose $C \in T$, $K \in C^2$, $K' \in C^2$, and $c \in \partial C$. Let i and i' be the ones of 0 and 1 with $c \in V_i(K)$ and $c \in V_{i'}(K')$ and take $j \neq i$ and $j' \neq i'$ in 2. If $K' \subset K$ or $K \subset V_{i'}(K')$ and $K' \subset V_i(K)$ then $(C - K^0) \subset [V_i(K) \cap V_{i'}(K')] \cup [V_j(K) \cap V_{j'}(K')]$.*

Proof. Without loss of generality $i = i' = 0$ and $j = j' = 1$, $\partial K = \{x_0, x_1\}$, $\partial K' = \{x'_0, x'_1\}$ and $\{W_0, W_1\}$ and $\{W'_0, W'_1\}$ are pairs of open sets with disjoint closures such that $x_k \in W_k$, $x'_k \in W'_k$, $V_k(K) = H^5(x_k, W_k)$ and $V_k(K') = H^5(x'_k, W'_k)$ for all $k \leqslant 2$. For all h and k in 2, let $V_{hk} = V_h(K) \cap V_k(K')$. So we want to prove that $(C - K^0) \subset (V_{00} \cup V_{11})$ in our two cases.

(a) Suppose $K' \subset K$. Then $K' \in K^2$. Since $c \in H(x_0, W_0) \cap H(x'_0, W'_0)$ and $W'_0 \cap W'_1 = \emptyset$ and $x_0 \in V_0(K') \cup V_1(K')$, $x_0 \in V_0(K')$. Suppose $(C - K^0) \not\subset (V_{00} \cup V_{11})$. If $x_1 \in V_0(K')$ then choose open U_0 and U_1 contained in $V_0(K')$ with disjoint closures such that $x_0 \in U_0$, $x_1 \in U_1$. Then $(*)$ for $K \in \mathcal{C}$ is contradicted by $H^3(x_0, U_0)$, $H^3(x_1, U_1)$ and $V_1(K')$. So $x_1 \in V_1(K')$.

Any point of $C - K^0$ not in $V_{00} \cup V_{11}$ must be in V_{01} or V_{10}, so suppose, for instance, that there is a point $p \in V_{01} \cap (C - K^0)$. But then the three points $p \in V_{01}$, $x_0 \in V_{00}$, and $x_1 \in V_{11}$ contradict $(**)$ for $K \in \mathcal{C}$ and the same is true if $p \in V_{10}$.

(b) Suppose $K \cap K' = \emptyset$, $K \subset \overline{V_0(K)}$ and $K' \subset \overline{V_0(K')}$. Since $x_1 \notin W'_1$ and $x'_1 \notin W_1$, $H(x_1 W_1) \cap H(x'_1 \cap W'_1) = \emptyset$ and $\overline{H^2(x_1, W_1)} \cap \overline{H^2(x_1, W'_1)} = \emptyset$. Since K and K' are in C^2, $H^3(x_1, H^2(x_1, W_1)) \not\subset C$ and $H^3(x'_1, H^2(x'_1, W'_1)) \not\subset C$ and $H^3(c, V_{00}) \not\subset C$. But these three facts contradict $(*)$ for $C \in \mathcal{C}$.

If $K \in T^2$, there is a minimal $\mu(K) \in \omega$ for which both $0 < \mu(K)$ and there is $M(K) \in T_{\mu(K)}$ with $K \in M(K)^2$. If $C \in T_\alpha$ for some $\alpha > 0$ and there is $K \in C^2$, choose $K(C) \in C^2$ such that $\mu(K(C))$ is minimal for all $K \in C^2$. Observe that Lemma 4 implies that for every $K \in C^2$, $M(K) - C^0$ intersects $V_0(K)$ in the same set it intersects one of $V_0(K(C))$ and $V_1(K(C))$ in. Since $C \in \mathcal{C}$, $|\partial C \cap V_0(K)| \leqslant 1$. By Lemma 1, if $p \in (M(K) - C^0)$, $p \in (M(K) - C^0) \cap V_0(K) \cap E$ for some $E \in (T_\alpha - \{C\})$, then $E \subset (M(K) - C^0) \cap V_0(K)$. Clearly $V_1(K)$ has similar properties. $\square$

4. The construction of Y_β and f_β

For each $\beta \leqslant \omega_1$ we shall inductively choose a linearly ordered space $\langle Y_\beta, \leqslant_\beta\rangle$ and a function f_β such that:

(1) $\langle Y_\beta, \leqslant_\beta\rangle$ is compact and f_β is a continuous map from Y_β onto $X_\beta = X - \bigcup\{D^0 \mid D \in T^*_\beta\}$. Since the terms of the T_α's are strictly decreasing as α increases, $X_{\omega_1} = X$; the existence of Y_{ω_1} and f_{ω_1} will thus prove Theorem 2.

(2) If $\alpha \leqslant \beta$, then $\langle Y_\alpha, \leqslant_\alpha\rangle \subset \langle Y_\beta, \leqslant_\beta\rangle$ and $f_\beta \restriction Y_\alpha = f_\alpha$.
If $Z \subset X$ we use $f_\beta^{-1}(Z)$ to mean $f_\beta^{-1}(Z \cap X_\beta)$ since the range of f_β is X_β.
If $x <_\beta y$ in $\langle Y_\beta, \leqslant_\beta\rangle$, by $(x, y)_\beta$ we mean the usual open interval of $\langle Y_\beta, \leqslant_\beta\rangle$ between x and y.

(3) If $0 < \beta$ and $K \in T_\beta$, there are adjacent $K^- <_\beta K^+$ in $\langle Y_\beta, \leqslant_\beta\rangle$ such that $\partial K \subset f_\beta(\{K^-, K^+\})$ and $\{\{K^-, K^+\} \mid K \in T_\beta\}$ are disjoint. Also, if, for $\gamma < \beta$, K_γ

is the term of T_γ containing K, then $f_\beta(K^-) \notin K$ implies there is $\delta < \beta$ with $f_\beta(K^-) = f_\gamma(K^-)$ for all $\delta \leqslant \gamma \leqslant \beta$.

(4) If $0 < \alpha < \beta$, $Y_\beta - Y_\alpha = \bigcup\{(C^-, C^+)_\beta \mid C \in T^*_\alpha\}$ and if $C \in T^*_\alpha$, $D \in T_\beta$, and $D \subset C$, $C^- \leqslant_\beta D^- <_\beta D^+ \leqslant_\beta C^+$.

(5) If $0 < \alpha < \beta$ and $C \in T^*_\alpha$, then $f_\beta^{-1}(C^0) \subset (C^-, C^+)_\beta \subset f_\beta^{-1}(C)$.

(6) If $0 < \alpha \leqslant \beta$, $C \in T^*_\alpha$, and $K \in C^2 \cap T_\tau$ for some $\beta < \tau < \omega_1$, then $[C, \beta, K]$ holds, where $[C, \beta, K]$ is the statement:

$$\begin{aligned}&\left[(C^-, C^+)_\beta \cap f_\beta^{-1}((C - K^0) \cap V_0(K))\right]\\ &\quad <_\beta \left[(C^-, C^+)_\beta \cap f_\beta^{-1}((C - K^0) \cap V_1(K))\right]\end{aligned}$$

or the same statement with $<_\beta$ replaced by $>_\beta$.

If $\beta = 0$, $X_\beta = \emptyset$ and we take $Y_0 = f_0 = \emptyset$.

Suppose $0 < \beta \leqslant \omega_1$ and Y_α and f_α as desired have been defined for all $\alpha < \beta$.

First assume β is a limit ordinal. Then $f^*_\beta = \bigcup_{\alpha<\beta} f_\alpha$ maps $\langle Y^*_\beta, \leqslant\rangle = \bigcup_{\alpha<\beta}\langle Y_\alpha, \leqslant_\alpha\rangle$ onto $X^*_\beta \subset X - \bigcup T_\beta$. Usually $\langle Y^*_\beta, \leqslant\rangle$ is not compact nor f^*_β continuous.

Suppose $D \in T_\beta$. Then $D = \bigcap_{\alpha<\beta} C_\alpha$ where C_α is the term of T_α containing D. Let $Y^-_D = \{y \in Y^*_\beta \mid \exists\alpha < \beta \text{ with } y \leqslant_\alpha C^-_\alpha\}$ and $Y^+_D = \{y \in Y^*_\beta \mid \exists\alpha < \beta \text{ with } y \geqslant_\alpha C^+_\alpha\}$.

Suppose $D^2 \neq \emptyset$ (which happens for instance if $|\partial D| = 2$). Pick $K \in D^2$. There is $\alpha < \beta$ with $K \in C^2_\alpha$. So we can assume that $[C_\delta, \gamma, K]$ holds for all $\alpha < \delta < \gamma < \beta$ with $<_\gamma$. (Otherwise we have $>_\gamma$ for all δ and γ.) Suppose there is δ with $\alpha < \delta < \beta$ such that $(C_\delta - D^0) \cap V_0(K) = \emptyset$. Then $f_\gamma(C^-_\delta, C^+_\gamma)_\gamma = \emptyset$ for all $\delta < \gamma < \beta$ and C^-_δ is maximal in $\langle Y^-_D, \leqslant\rangle$ and we define $D^- = C^-_\delta = C^-_\gamma$ for all $\delta < \gamma < \beta$. Otherwise we just name some totally new object D^- and define $f_\beta(D^-)$ to be the unique point in $\partial D \cap V_0(K)$. Similarly, if there is a minimal z in $\langle Y^+_D, \leqslant\rangle$ we define $D^+ = z$ while otherwise we just name some totally new object D^+ and define $f_\beta(D^+)$ to be the unique point in $\partial D \cap V_1(K)$. If $>_\gamma$ holds for all $\alpha < \gamma < \beta$, make the same definitions with $V_0(K)$ and $V_1(K)$ interchanged.

If $D^2 = \emptyset$, arbitrarily choose two new objects D^- and D^+ and define $f_\beta(D^-) = f_\beta(D^+)$ to be the unique point of ∂D.

Define

$$Y_\beta = Y^*_\beta \cup \left\{D^- \mid D \in T_\beta\right\} \cup \left\{D^+ \mid D \in T_\beta\right\}.$$

We have defined $f_\beta(D^-)$ and $f_\beta(D^+)$ for D^- and D^+ not in Y^*_β, and we extend to $f_\beta : Y_\beta \to X_\beta$ by requiring that $f_\beta \restriction Y^*_\beta = f^*_\beta$.

It remains to define the total order $\leqslant_\beta$ on Y_β in such a way that our induction hypotheses hold. For $a \neq z$ in Y_β define $a <_\beta z$ if

(1) $a < z$ in $\langle Y^*_\beta \leqslant\rangle$, or

(2) $a = D^-$ and $z = D^+$ for the same $D \in T^*_\beta$, or

(3) a is D^- or D^+ for some $D \in T^*_\beta$ having $z \in Y^+_D$, or

(4) z is D^- or D^+ for some $D \in T^*_\beta$ having $a \in Y^-_D$, or

(5) for some $D \neq E$ in T_β, a is D^- or D^+, z is E^- or E^+, $Y^-_D \subset Y^-_E$, and $Y^+_E \subset Y^+_D$.

With this definition it is easy to check that our induction hypotheses (2)–(6) are satisfied. To see that a subset of Y_β, has a least upper bound (and similarly a greatest lower bound) in $\langle Y_\beta, \leqslant_\beta\rangle$ making Y_β compact, one only needs to observe that if its restriction to $\langle Y^*_\beta, \leqslant\rangle$ did not already have one, we added one (or two) to Y_β. It remains to check that f_β is continuous.

To this end suppose $y \in Y_\beta$, $f_\beta(y) = p$, and $p \in V$ which is open in X. Our aim is to find an open interval I of $\langle Y_\beta, \leqslant_\beta\rangle$ containing y with $f_\beta(I) \subset V$.

By the saturation lemma, there is an open neighborhood Z of p, $Z \subset V$, such that $p \notin C$, $C \in T$, and $C \cap Z \neq \emptyset$, imply $C \subset V$.

Suppose there is $\alpha < \beta$ such that $y \in Y_\alpha$ but y is neither C^- nor C^+ for any $C \in T_\alpha$. By the continuity of f_α, there are $y^- <_\alpha y <_\alpha y^+$ in $\langle Y_\alpha, \leqslant_\alpha\rangle$ such that $f_\alpha((y^-, y^+)_\alpha) \subset Z$. Let

$$\mathcal{A} = \left\{C \in T_\alpha \mid y^- \leqslant_\alpha C^- <_\alpha C^+ \leqslant_\alpha y^+\right\}.$$

Since $y^- = C^-$ implies $C^+ <_\alpha y$ we can assume $y^- <_\alpha C^-$ and, similarly, $C^+ <_\alpha y^+$ for all $C \in \mathcal{A}$. Also, if there were $A \in \mathcal{A}$ with $p \in A$, there is only one such A and $y \notin \{A^-, A^+\}$. So we can assume $\{A^-, A^+\} \cap (y^-, y^+)_\alpha = \emptyset$. Thus if $C \in \mathcal{A}$, $p \notin C$. But, if $C \in \mathcal{A}$, either $f_\alpha(C^-)$ or $f_\alpha(C^+)$ is in ∂C and thus in Z; so $C \cap Z \neq \emptyset$. By our choice of Z then, $C \subset V$ if $C \in \mathcal{A}$. By (5),

$$(C^-, C^+)_\beta \subset f_\beta^{-1}(C \cap X_\beta) \quad \text{and}$$
$$(y^-, y^+)_\beta = (y^-, y^+)_\alpha \cup \bigcup\left\{(C^-, C^+)_\beta \mid C \in \mathcal{A}\right\}.$$

Thus $I = (y^-, y^+)_\beta$ is an interval of $\langle Y_\beta, \leqslant_\beta\rangle$ to which y belongs and $f_\beta(I) \subset V$ as desired.

Observe that if there are a cofinal S in β and for each $\alpha \in S$ some $D_\alpha \in T_\alpha$ such that $y \in \{D^-_\alpha, D^+_\alpha\}$, then we can assume that $y = D^-_\alpha$ for all $\alpha \in S$, or $y = D^+_\alpha$ for all $\alpha \in S$. Then by (5), $\alpha < \alpha'$ in S implies $D_\alpha \supset D_{\alpha'}$. If $D = \bigcap_{\alpha \in S} D_\alpha$, by definition, $y = D^-$ for this $D \in T_\beta$, or $y = D^+$.

Thus we can assume there is $D \in T_\beta$ such that y is D^- (we could equally well say D^+). Now we have two cases, namely, $p \in D$ or $p \notin D$. For $\alpha < \beta$ define D_α to be the unique member of T_α with $D \subset D_\alpha$.

If $p \notin D$, y is not only D^- but there is $\alpha < \beta$ such that $p \notin D_\alpha$ while $y = D^-_\alpha$. Since f_α is continuous there is $y^- <_\alpha y$ such that $f_\alpha((y^-, D^+_\alpha)_\alpha) \subset Z$. Let $\mathcal{A} = \{C \in T^*_\alpha \mid y^- \leqslant_\alpha C^- <_\alpha C <_\alpha y\}$. Again we can assume that for all $C \in \mathcal{A}$, $y^- <_\alpha C^-$ and hence $C \cap Z \neq \emptyset$ and $p \notin C$, so $C \subset V$. But $(y^-, D^+)_\beta = (y^-, D^+_\alpha)_\alpha \cup \bigcup\{(C^-, C^+)_\beta \mid C \in \mathcal{A}\}$. Thus $I = (y^-, D^+)_\beta$ is an open interval in $\langle Y_\beta, \leqslant_\beta\rangle$ with $y \in I$ and $f_\beta(I) \subset V$.

So we can suppose $p \in D$ (and $y = D^-$). If $|\partial D| = 2$, then $f_\beta(y)$ is in $V_0(D)$ or $V_1(D)$, say $V_0(D)$. There is $\alpha < \beta$ such that $D \in D^2_\alpha$ and $(D_\alpha - D) \cap V_0(D) \subset V$. By (6)

$$\left[(D^-_\alpha, D^+_\alpha)_\beta \cap f_\beta^{-1}((D_\alpha - D^0) \cap V_0(D))\right] <_\beta D^- = y <_\beta D^+.$$

Here then $I = (D^-_\alpha, D^+)_\beta$ has $y \in I$ and $f_\beta(I) \subset V$. Still assuming $p \in D$ and $y = D^-$, suppose $\partial D = \{p\}$. Choose $\alpha < \beta$ so $(D_\alpha - D) \subset V$. Then $I = (D^-_\alpha, D^+)_\beta$ again has $y \in I$ and $f_\beta(I) \subset V$.

This completes the proof that f_β is continuous for limit β.

Now assume that $\beta = \alpha + 1 \leqslant \omega_1$. We must define $\langle Y_\beta, \leqslant_\beta \rangle$ and f_β satisfying our 6 induction hypotheses.

Suppose $C \in T^*_\alpha$ and define $\mathcal{D} = \{D \in \mathcal{K}(C) \mid \exists K_D \in D^2 \cap C^2\}$. We wish to define a totally ordered set $\langle \mathcal{E}, \leqslant_E \rangle$ of subsets of $C \cap X_\beta$.

If $\mathcal{D} = \emptyset$, let $\mathcal{E} = \{C \cap X_\beta\}$ and let $\leqslant_{\mathcal{E}}$ be the trivial order on this one element set. This is the case if $\beta = 1$ since $T_\emptyset = X$ and $X^2 = \emptyset$ but may also be true for other C.

Suppose $\mathcal{D} \neq \emptyset$ and hence that $\alpha > 0$ and that C^- and C^+ have been defined. For each $D \in \mathcal{D}$ we choose i_D and j_D from (2) so that $f_\alpha(C^-) \in V_{i_D}(K_D)$ if $f_\alpha(C^-) \in C$, and $f_\alpha(C^+) \in V_{j_D}(K_D)$ otherwise. Then choose the unchosen i_D or j_D so $i_D \neq j_D$. Since $C \in \mathcal{C}$, if $|\partial C| = 2$ and one member of ∂C is $f_\alpha(C^-)$ and is in $V_0(K_D)$ and the other is $f_\alpha(C^+)$ and is in $V_1(K_D)$. If $D \in \mathcal{D}$ let $A_D = (C - D^0) \cap V_{i_D}(K_D)$ and $Z_D = (C - D^0) \cap V_{j_D}(K_D)$. If $B \in \mathcal{K}(C) - \{D\}$ either $B \subset A_D$ or $B \subset Z_D$. If $B \in \mathcal{D} - \{D\}$, then $F \subset A_D$ implies $D \subset Z_B$ and $B \subset Z_D$ implies $D \subset A_B$. To see this suppose $f_\alpha(C) = c \in \partial C$. Then $c \in A_D \cap A_B$. Either $x_{i_D} \in V_{i_B}(K_B)$ or $x_{i_B} \in V_{i_D}(K_D)$ so either $B \subset A_D$ or $D \subset A_B$. But by Lemma 4 we cannot have both since $x_{j_B} \in (C - D^0) - (V_{i_B}(K_B) \cup V_{j_D}(K_D))$.

For $x \in C \cap X_\beta$, let $E(x) = \{p \in (C \cap X_\beta) \mid \forall D \in \mathcal{D},\ p \in A_D$ if and only if $x \in A_D\}$. Observe that $\mathcal{E} = \{E(x) \mid x \in (C \cap X_\beta)\}$ is a disjoint compact cover of $C \cap X_\beta$. Define $E(x) <_{\mathcal{E}} E(y)$ if there is some $D \in \mathcal{D}$ with $y \in Z_D$ and $x \in A_D$. If $f_\alpha(C^-) \in \partial C$, then $f_\alpha(C^-) \in A_D$ for all $D \in \mathcal{D}$ and otherwise $f_\alpha(C^+) \in Z_D$ for all $D \in \mathcal{D}$. Thus $\langle \mathcal{E}, \leqslant_{\mathcal{E}} \rangle$ is a totally ordered set.

Suppose $E \in \mathcal{E}$. Since all subsets of separable, monotonically normal spaces have these properties [2], and $E \subset C - \bigcup\{K^0 \mid K \in \mathcal{K}(C)$ and $K \neq K^0\}$ which is separable, E is a compact, separable, monotonically normal space and by Theorem 1, there is a compact linearly ordered space $\langle L, \leqslant_L \rangle$ and continuous map f_L from L onto E. There is at most one point a of E which either is in $\partial D \cap Z_D$ for some $D \in \mathcal{D}$ or is a limit of $\bigcup\{A_D \mid D \in \mathcal{D}$ with $E \subset Z_D\}$. (See Lemma 6 of [5] on p. 403 for a proof that such an increasing union can have at most one limit point in a monotonically normal space.) If such an a exists we can assume that $a = f_L$ (the first point of $\langle L, \leqslant_L \rangle$). (To achieve this just add some new object as a first point of L and map it onto a; the new $\langle L, \leqslant_L \rangle$ is still compact and the new f_L still continuous and onto.) If $a \in D \in \mathcal{D}$, define $D^+ =$ the first point of $\langle L, \leqslant_L \rangle$. Similarly, if $z \in E$ either is in $\partial D \cap A_D$ for some $D \in \mathcal{D}$ or is the unique limit point of $\bigcup\{Z_D \mid D \in \mathcal{D}$ and $E \subset A_D\}$, we choose $\langle L, \leqslant_L \rangle$ so z is f_L (the last point of $\langle L, \leqslant_L \rangle$). Observe that $\partial D \cap A_D \cap Z_D = \emptyset$. And if such a $z \in D \in \mathcal{D}$, we define $D^- =$ the last point of $\langle L, \leqslant_L \rangle$.

Still keeping E fixed, for each $K \in \mathcal{K}(C) - \mathcal{D}$ with $\partial K \subset E$, choose $y_K \in f_L^{-1}(\partial K)$ and two new objects K^- and K^+. Let $Y_E = L \cup \bigcup\{\{K^-, K^+\} \mid K \in K(C) - \mathcal{D}$ and $\partial K \subset E\}$ and let $\leqslant_E$ be the total ordering of Y_E which extends $\langle L, \leqslant_L \rangle$ such that, for each $K \in \mathcal{K}(C) - \mathcal{D}$ with $\partial K \subset E$, $y_K <_E K^- <_E K^+$ is an interval of $\langle Y_E, \leqslant_E \rangle$. Define $f_E : Y_E \to E$ by $f_E \restriction L = f_L$ and, for $K \in \mathcal{K}(C) - \mathcal{D}$ with $\partial K \subset E$, $f_E(K^+) = f_L(y_K)$, $f_E(K^-) = f_L(y_K)$ if $|\partial K| = 1$, and $f_E(K^-) =$ the point of ∂K different from $f_L(y_K)$ if $|\partial K| = 2$. If one replaces some of the points of a linearly ordered compact space, $\langle L, \leqslant_L \rangle$ in this case, by three points, the result, $\langle Y_E, \leqslant_E \rangle$, is still linearly ordered and compact. Since the end points of our three-point-sets have the same image, our new function, f_E, is still continuous by the saturation lemma.

Now just keeping C fixed, let $M = \bigcup\{Y_E \mid E \in \mathcal{E}\}$. We totally order M by $\leqslant_M$, defined by $x <_M y$ in M if $x \in Y_E$ and $y \in Y_{E'}$, for some E and E' in $\mathcal{E}$, and either $E <_{\mathcal{E}} E'$ or $E = E'$ and $x <_E y$. Define $f_M : M \to C \cap X_\beta$ by $f_M(y) = f_E(y)$ for all $y \in Y_E$. By our choice for each $E \in \mathcal{E}$ of the first and last points of its compact $\langle L, \leqslant_L\rangle$ which is preserved in $\langle Y_E, \leqslant_E\rangle$, the continuity of f_E and compactness of $\langle Y_E, \leqslant_E\rangle$ forces the compactness of $\langle M, \leqslant_M\rangle$ and continuity of f_M onto $C \cap X_\beta$.

If $\beta = 1$, define $Y_1 = \langle M, \leqslant_M\rangle$ and $f_1 = f_M$. Note that $\mathcal{E} = \{X_1\}$ so $\langle M, \leqslant_M\rangle = \langle Y_{X_1}, \leqslant_{X_1}\rangle$ and $f_M = f_{X_1}$ and the induction hypotheses are satisfied. So assume $\beta > 1$ (and thus C^- and C^+ are defined).

Let

$$\mathcal{D}^- = \bigl\{D \in \mathcal{D} \mid D^- \text{ is not defined but } V_{i_D}(K_D) \cap C \cap X_\beta \neq \emptyset\bigr\} \quad \text{and}$$
$$\mathcal{D}^+ = \bigl\{D \in \mathcal{D} \mid D^+ \text{ is not defined but } V_{j_D}(K_D) \cap C \cap X_\beta \neq \emptyset\bigr\}.$$

For each $D \in \mathcal{D}^-$ choose a new object D^- and for each $D \in \mathcal{D}^+$ choose a new object D^+. If $D \in \mathcal{D}^-$, let $D' = \sup\{y \in M \mid y <_M D^+\}$; D' exists since $V_{i_D}(K_D) \cap C \cap X_\beta \neq \emptyset$ and $\langle M, \leqslant_M\rangle$ is compact. Similarly, let $D' = \inf\{y \in M \mid y >_M D^-\}$ if $D \in \mathcal{D}^+$.

Let $Y_C = M \cup \{C^-, C^+\} \cup \{D^- \mid D \in \mathcal{D}^-\} \cup \{D^+ \mid D \in \mathcal{D}^+\}$. Let $\leqslant_C$ be the total ordering of Y_C which extends $\langle M, \leqslant_M\rangle$, has C^- as its first and C^+ as its last point, and, for all $D \in \mathcal{D}^- \cup \mathcal{D}^+$ has $D^- < D^+$ as an interval of $\langle Y_C, \leqslant_C\rangle$. Since $\langle M, \leqslant_M\rangle$ is compact, so is $\langle Y_C, \leqslant_C\rangle$. Define $f_C : Y_C \to f_\alpha(\{C^-, C^+\}) \cup (C \cap X_\beta)$ to be the map extending f_M and $f_\alpha \restriction \{C^-, C^+\}$ with $f_C(D^-) = f_M(D')$ if $D \in \mathcal{D}^-$ and $f_C(D^+) = f_M(D')$ if $D \in \mathcal{D}^+$. Since f_M is continuous, so is f_C.

There is at most one $D \in \mathcal{D}$ such that $V_{i_D}(K_D) \cap C \cap X_\beta = \emptyset$ and in this case we define D^- (which is so far undefined) to be C^-; in this case D^- and D^+ are adjacent in $\langle Y_C, \leqslant_C\rangle$. Similarly if $V_{j_D}(K_D) \cap C \cap X_\beta = \emptyset$ for some $D \in \mathcal{D}$, define $D^+ = C^+$. This completes our choice of K^- and K^+ for all $K \in \mathcal{K}(C)$ with $C \in T^*_\alpha$ and thus for all $K \in T_\beta$.

Define $Y_\beta = Y_\alpha \cup \{Y_C \mid C \in T^*_\alpha\}$ and let $\leqslant_\beta$ be the total order on Y_β which extends $\leqslant_\alpha$ and has each $\langle Y_C, \leqslant_C\rangle$ as an interval of $\langle Y_\beta, \leqslant_\beta\rangle$. Since C^- and C^+ are adjacent in $\langle Y_\alpha, \leqslant_\alpha\rangle$ and the first and last points of $\langle Y_C, \leqslant_C\rangle$, the order is well defined and compact. Define $f_\beta : Y_\beta \to X_\beta$ as the map extending f_α on Y_α and extending f_C and Y_C for all $C \in T^*_\alpha$; again $Y_\alpha \cap Y_C = \{C^-, C^+\}$ and f_α and f_C agree on these points. So the continuity of f_α and of the f_C's ensures the continuity of f_β since such insertions preserve linear order, compactness, and continuity.

One easily checks that conditions (3)–(6) of our induction hypothesis are satisfied by this choice of $\langle Y_\beta, \leqslant_\beta\rangle$ and f_β (and K^- and K^+ for $K \in T_\beta$). Thus the proof of the theorem is complete. $\square$

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